PARAMETERIZATION OF MODEL VALIDATING SETS FOR UNCERTAINTY BOUND OPTIMIZATIONS

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Abstract

Given experimental data and *a priori* assumptions on nominal model and a linear fractional transformation uncertainty structure, feasible conditions for model validation is given. All unknown but bounded exogenous inputs are assumed to occur at the plant outputs. With the satisfaction of the feasible conditions for model validation, it is shown that a parameterization of all model validating sets of plant models is possible. The new parameterization can be used as a basis for the development of a systematic way to construct model validating uncertainty models which have specific linear fractional transformation structure for use in robust control design and analysis. The proposed feasible condition (existence) test and the parameterization is computationally attractive as compared to similar tests currently available.

1 Introduction

1.1 Motivation

In applying multivariable robust control analysis and synthesis techniques to linear, time-invariant systems, as in for example [1], a "robust control design model" (a particular set of plant models described by a nominal model and norm bounded model uncertainty and exogenous inputs) is required *a priori*. Nominal models are usually associated with a single "best" model, although what is considered "best" is debatable. Mathematical models derived from first principles are typically used as nominal models or sometimes identified from system identification experiments. In some cases where the physical conditions are not accurately or reliably known due to causes unknown or when simple models are desirable, it makes sense to require that the set of plants in question at least satisfy model validation conditions in the frequency domain [2] and/or time domain [3] with respect to available measurement data.

1.2 Relation to Earlier Work

There exists a wealth of literature related to model validation, uncertainty modeling and identification for control (see for example the Special Issue on System Identification and Control in IEEE Transactions [4]). Unfortunately, it appears that there is not a systematic methodology that can be demonstrated to work for a significant class of problems. For brevity, we highlight only closely related earlier work to our approach. Indeed, our approach is basically an extension of a special case formulated in [2]. This special case involves the assumption that all exogenous inputs occur at the plant output.

The basic question posed in [2] to [6] is: given a robust control design model, does a plant exists within this set which will reproduce a given input and output measurements? This model validation question can be viewed as a check on two properties. First, the choice of uncertain parameters and the corresponding postulated uncertainty connection leading to the Linear Fractional Transformation (LFT) structure must be *sufficiently rich* to admit a perturbed model (however large the uncertainty bound must be) about the given nominal that will faithfully reproduce the observed output data from the given input data. Second, assuming the first property is satisfied, the postulated norm bound on the model uncertainty and exogenous disturbance must be *sufficiently large* to admit a model validating plant.

The viewpoint taken in this paper with regards to model validation is to focus on the first property,

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i.e., whether an *a priori* given LFT structure for an uncertainty model with some noise allowance can lead to a model validating set. In the seminal work in [2], the first property is referred to as the "feasible" condition. The rationale for the above viewpoint is twofold: (1) the model validation question becomes obviously simpler to address, and (2) once a feasible interconnection structure for uncertainty is found, a model validating set can always be constructed, using the results in this paper. Hence, our viewpoint differs from the more prevalent binary test for model validation given both interconnection structure and *a priori* uncertainty bounds. Our novel viewpoint also alleviate the dilemma of what to change if a model validation test fails.

In an earlier attempt to obtain a simpler problem formulation and solution than the approach taken by [2] to [6], [7] considered a special case where all exogenous inputs are either known or are very small and occurs only at the output. This simpler formulation has been extended to closed loop systems in [8] and subsequently applied to a challenging experimental testbed with very encouraging results [9]. Recognizing the dependence of minimum norm model validating uncertainty bounds on the input directions for multivariable uncertainties, [10] formulated a min-max problem to address this. Although the approach taken in [7] to [10] appears to work reasonably well for problems with an arbitrary number of structured full complex blocks only, it became clear through applications that problems with parametric (and often repeated) uncertainties gave unsatisfactory results. This was somewhat expected because the additional structure in the repeated scalar uncertainties were not incorporated in the original problem formulation. Hence, the most recent work reported in [11] extends the previous approach to include repeated scalar parametric uncertainties along with an arbitrary number of full complex blocks. This paper provides the detail proofs omitted in [11], expands the uncertainty set to include unknown but bounded output noise, and ultimately parameterize all model validating sets for a fixed LFT structure.

1.3 Outline of Paper

In Section 2, a problem definition is given whereby uncertainty bounds are viewed as bounds on fictitious uncertainty signals which satisfies $P-\Delta$ transmission conditions while resulting in zero output errors. In Section 3 we derive existence conditions for model validation followed by parameterizing all model validating uncertainty sets with respect to *a priori* assumptions on the noise and uncertainty structure. In Section 4, we briefly outline the possibilities in utilizing the parameterizations given in Section 3. Section 5 is a summary of results and final remarks.

2 Problem Definition

For a given physical system which happens to be approximately linear and time-invariant, suppose we have measurements of its inputs, u, and the outputs, y, (see Figure 1). The corresponding model output,

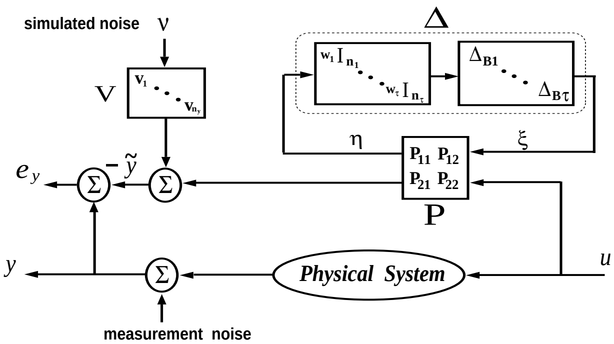

Figure 1: Block diagram for open loop robust ID

$\tilde{y}$, is a sum of simulated noise through a noise filter, V, and the output from an upper LFT model, $F_u(P,\Delta)$, which depends on an augmented plant, P, and structured uncertainty, $\Delta \in \mathcal{D}$, where

$$\mathcal{D} := \{\Delta \in \mathcal{C}^{m\times n} : \ \Delta = \text{diag}(\Delta_1, \ldots, \Delta_\tau), \ \Delta_i \in \mathcal{C}^{m_i \times n_i}\} \tag{1}$$

and τ denotes the number of uncertainty blocks.

Suppose the measurements are taken in the discrete-time domain and consider a discrete frequency domain formulation, so called Constant Matrix case in [2]. For simplicity, we assume that a discrete Fourier transform has been performed and do not consider the additional affects of realistic signal conditioning operations typically performed on the raw discrete-time signals (see for example [12]).

Since we are primarily concerned with the size of the uncertainty blocks, note from Figure 1 that an uncertainty bound in terms of its maximum singular value can be written as a ratio of norms

$$\bar{\sigma}(\Delta) := \sup_{\eta'} \frac{||\Delta\eta'||}{||\eta'||} \geq \frac{||\xi||}{||\eta||} \tag{2}$$

where $||\cdot||$ represent the Euclidian norm of its vector argument. For uncertainty bounds on components, let $\text{col}(\xi_1, \cdots, \xi_\tau)$ and $\text{col}(\eta_1, \cdots, \eta_\tau)$ be the partitioning of the vectors ξ and η which conform to the block diagonal partition of Δ in Equation (1). Then, since

$$\bar{\sigma}(\Delta) = \max_{1 \leq i \leq \tau} \bar{\sigma}(\Delta_i), \tag{3}$$

it is of interest to note that

$$\bar{\sigma}(\Delta_i) := \sup_{\eta_i'} \frac{||\Delta_i \eta_i'||}{||\eta_i'||} \geq \frac{||\xi_i||}{||\eta_i||} \qquad 1 \leq i \leq \tau. \tag{4}$$

Of course fictitious signals ξ and η cannot be measured nor are they arbitrary so that it is necessary to look at their dependence on real signals u and y and their transmission through a postulated system P at each frequency. The signals, ξ and η, whose norm ratios determine the uncertainty sizes must be consistent with their transmission through the augmented plant and also reproduce the measured inputs and outputs with some help from simulated output noise.

The output error is given by

$$e_y := y - \tilde{y} = e_y^o - V\nu - P_{21}\xi \tag{5}$$

where $e_y^o := y - P_{22}u$ denotes the nominal output error due to the nominal plant. The terms, $V\nu$ and $P_{21}\xi$ in Equation (5) represent the uncertainty freedoms in an attempt to negate or "explain" the effect of nominal output error. These two terms correspond to simulated measurement noise at output and LFT structured uncertainty model.

Definition (Model Validation [2]):

Given measurements of the input and output signals, u and y, a noise filter, V, an augmented nominal plant model, P, and a matrix of structured uncertainty norms, W, the set of plants (robust control design models)

$$\mathcal{S}_W = \{\Delta \in \mathcal{D} \ : \ \Delta = W\Delta_B, \bar{\sigma}(\Delta_B) \leq 1\}$$

is said to be a model validating set (of plant models) if it contains a plant Δ such that there exists an error signal, ν, with $||\nu|| \leq 1$ for which

$$y = F_u(P, \Delta)u + V\nu. \tag{6}$$

Notice that in the above definition, W denotes the radii of the structured uncertainty unit ball as defined by $\bar{\sigma}(\Delta_B) \leq 1$. In short, a set of plants is model validating if it can reproduce the given measurements while subject to *a priori* constraints. Of course, as noted earlier [2]-[6], one can never really "validate" a model since fresh data could potentially invalidate it.

In the sequal, we seek to characterize model validation in a way that will allow a convenient parameterization of all model validating uncertainty sets with respect to available input/output measurement for a given LFT structure. The idea is that ultimately, the controls engineer will be able to interactively shape the model validating uncertainty weights based on his controller design and analysis results. To this end, we first investigate a feasibility condition (or necessary condition) for model validation. At each frequency, is there a pair (ν, ξ) which makes the output error in Equation (5) zero? Note that once P, u, and ξ are specified, η is completely determined. The next step is to incorporate the constraints due to *a priori* structure in the uncertainties which may limit the feasible (ξ, η) signals. This leads to necessary and sufficient conditions for model validation. The final step is to parameterize all model validating sets of plant models.

It is clear that if the noise vector ν is not restricted by a fixed bound, then any output residual can be zeroed out (without any help from Δ) if the noise filter V is non-singular. In this paper we assume that the noise filter V is given (as part of the *a priori* model assumption or a reasonable model determined from earlier system identification experiments) and the noise vector at each frequency is norm bounded by 1. This output noise model can be viewed as a model of a broad band exogenous noise typified by sensor noise. However, V, when judiciously chosen can reflect *a priori* bounds on the noise intensity or power spectrum of the unknown exogenous signal over a bandwidth of interest.

3 Parameterization of Model Validating Sets

In this section, we develop a theory to effectively parameterize all model validating uncertainty sets. We begin by first addressing the question: when does there exist a norm bounded or "admissible" noise ν with $||\nu|| \leq 1$ and a $\xi \in \mathcal{C}^m$ such that $e_y = 0$? At this point, notice that ξ is not required to be limited by any given bound. To answer the above question we first state a lemma, set some notation, and make some observations.

Lemma 1:

Let A be a matrix whose singular value decomposition (SVD) is

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix}, \tag{7}$$

where $U = [U_1 \ U_2]$ and $V = [V_1 \ V_2]$ are unitary, Σ_1 is diagonal and nonsingular, and the block matrix partitionings are conformal. Let b be a vector and c

a non-negative real constant. Then the inequality

$$||Ax+b|| \leq c \tag{8}$$

has a solution x if and only if

$$||U_2^H b|| \leq c. \tag{9}$$

Then the general solution to the inequality (8) is parameterized by

$$\begin{aligned} x &= V_1 y + V_2 z \quad (10) \\ y &= \Sigma_1^{-1}(w - U_1^H b) \quad (11) \end{aligned}$$

where z is arbitrary, and w is any vector with

$$||w|| \leq \sqrt{c^2 - ||U_2^H b||^2}. \tag{12}$$

Proof of Lemma 1:

Note that x can always be written as $V_1 y + V_2 z$ by taking $y = V_1^H x$ and $z = V_2^H x$. Now, from SVD of A, $AV_2 z = 0$, and $AV_1 y = U_1 \Sigma_1 y$ Therefore, for arbitrary y and z, if $x = V_1 y + V_2 z$ then $||Ax+b||^2 \leq c^2$ if and only if $||U_1 \Sigma_1 y + b||^2 \leq c^2$. Since U is unitary, multiplying by U^H preserves norm, so the last inequality is equivalent to

$$\left\| \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} U_1 \Sigma_1 y + \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} b \right\|^2 \leq c^2. \tag{13}$$

In its turn, this is equivalent to

$$\left\| \begin{matrix} \Sigma_1 y + U_1^H b \\ U_2^H b \end{matrix} \right\|^2 \leq c^2. \tag{14}$$

This can be rewritten as

$$||\Sigma_1 y + U_1^H b||^2 \leq c^2 - ||U_2^H b||^2. \tag{15}$$

So on the one hand, if inequality (8) has a solution, then that solution fits the parameterization given in Equations (10) – (12) and condition (9) is satisfied. On the other hand, if condition (9) is satisfied and first w and z and then y and finally x are picked according to the parameterization given in Equations (10) – (12), then this x provides a solution to inequality (8). $\square$

If we set $M = [P_{21}, V]$, then the condition $e_y = 0$ can be written as

$$M \left\{ \begin{matrix} \xi \\ \nu \end{matrix} \right\} = e_y^o. \tag{16}$$

This provides the first necessary condition for the existence of ν with $||\nu|| \leq 1$ and $\xi \in \mathcal{C}^m$ for which $e_y = 0$; namely:

$$e_y^o \in \mathrm{Im}(M) \tag{17}$$

Observe that if V is invertible, then condition (17) is true, since M has full row rank. Physically, this means that if the noise model at output is allowed to influence all output channels, then any output signal can be validated (without any help from Δ) if the noise is not constrained by a bound. Also, condition (17) is sufficient for the existence of $\nu \in \mathcal{C}^{n_y}$ and $\xi \in \mathcal{C}^m$ for which $e_y = 0$. The remainder of this discussion is aimed at finding a condition to insure that $||\nu|| \leq 1$.

If condition (17) is satisfied, then Equation (16) is solvable, and a complete parameterization of the solutions is given by

$$\left\{ \begin{matrix} \xi \\ \nu \end{matrix} \right\} = M^+ e_y^o + N_M \theta, \tag{18}$$

where N_M is a matrix whose columns form a basis for Ker(M), and the parameter θ is arbitrary. The notation $(\cdot)^+$ denotes the Moore-Penrose pseudo-inverse of $(\cdot)$.

If the subscripts $(\cdot)_\xi$ and $(\cdot)_\nu$ denote the row partitioned components of $(\cdot)$ corresponding to the ξ and ν vectors, respectively, in Equation (18) or any equation defining $\left\{ \begin{smallmatrix} \xi \\ \nu \end{smallmatrix} \right\}$, then Equation (18) implies that

$$\nu = (M^+)_\nu e_y^o + (N_M)_\nu \theta \tag{19}$$

and the problem is to characterize those θ for which $||\nu|| \leq 1$.

This can be done by a direct application of Lemma 1. First perform an SVD of $(N_M)_\nu$ as in Lemma 1:

$$(N_M)_\nu = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} \tag{20}$$

Then, by Lemma 1, Equation (19) can be solved for ν with $||\nu|| \leq 1$ if and only if

$$||T_2^H (M^+)_\nu e_y^o|| \leq 1. \tag{21}$$

Then, the general solution to Equation (19) is given by

$$\begin{aligned} \theta &= U_1 \gamma + U_2 \psi \quad (22) \\ \gamma &= \Sigma_1^{-1}(\phi - T_1^H (M^+)_\nu e_y^o) \quad (23) \end{aligned}$$

where ψ is arbitrary, and ϕ is any vector with

$$||\phi|| \leq b_o := \sqrt{1 - ||T_2^H (M^+)_\nu e_y^o||^2}. \tag{24}$$

In light of the definition in Equation (24), the necessary condition in (21) implies $b_o \geq 0$.

Lemma 2:

With the context and notation established in the previous paragraphs, there exists an uncertainty signal ξ which, when combined with the given input u and the nominal model P, produces an output which is within the noise allowance of the observed output y if and only if conditions (17) and (21) hold. If these conditions are satisfied, then all ξ are given by the parameterization

$$\xi = \xi_o + [A, \;\; B] \left\{ \begin{array}{c} \phi \\ \psi \end{array} \right\} \tag{25}$$

where $A := (N_M)_\xi U_1 \Sigma_1^{-1}$, $B := (N_M)_\xi U_2$, $\xi_o = [I, \;\; -AT_1^H] M^+ e_y^o$, ψ is arbitrary, and ϕ satisfies inequality (24).

Proof of Lemma 2:

The existence of ξ with the desired properties is equivalent to the solvability of Equation (16) with $\|\nu\| \leq 1$. We have already seen that Equation (16) can be solved with $\|\nu\| \leq 1$ if and only if conditions (17) and (21) hold, and all such solutions are parameterized by Equations (22) – (24).

When a value of θ as given by the parameterization in Equations (22) – (24) is substituted in Equation (18), it gives ξ as

$$\xi = (M^+)_\xi e_y^o + (N_M)_\xi \theta. \tag{26}$$

Using Equations (22) and (23) to eliminate θ and then γ from this expression for ξ results in the expression for ξ given Equation (25). □

Note that the existence aspect of Lemma 2 is also given earlier in [2] in a more general context. Lemma 2 gives a test resulting in either a yes or no answer. It is only concerned with testing the richness of the *a priori* LFT uncertainty structure and chosen levels of measurement noise against a given set of measured input and output data. Whether such ξ can be generated through the LFT uncertainty and how large it must be remains to be seen. If the test in Lemma 2 fails, then the model is invalidated either due to overly restricted levels of noise and/or insufficiently rich uncertainty LFT structure. What course of action to take if Lemma 2 test fails is not considered in this paper.

Suppose the above test in Lemma 2 passes. Indeed, Lemma 2 gives a parameterization of the set of all ξ that produces zero output error. That ξ be given by this parameterization provides a necessary conditions that ξ be a signal in a model validated robust control design model. For sufficiency, ξ must also satisfy the $P - \Delta$ feedback conditions

$$\xi = \Delta \eta \tag{27}$$

$$\eta = P_{11} \xi + P_{12} u \tag{28}$$

Since η can be readily computed from Equation (28) for a given ξ, we group Equations (25) and (28) as follows:

$$\left\{ \begin{array}{c} \eta \\ \xi \end{array} \right\} = \left\{ \begin{array}{c} \eta_o \\ \xi_o \end{array} \right\} + \left[\begin{array}{c} P_{11} \\ I \end{array} \right] [A, \;\; B] \left\{ \begin{array}{c} \phi \\ \psi \end{array} \right\} \tag{29}$$

Here, $\eta_o := P_{11} \xi_o + P_{12} u$, and the norm of ϕ is subject to condition (24). Equation (29) characterizes the set of all (ξ, η) vectors that produces zero output error. Of course, this set may be further constrained by the uncertainty structure given by Equation (27), which motivates the next lemma.

Consider a basic fact from linear algebra as noted earlier in [7]:

Lemma 3:

If $u \in \mathcal{C}^m$, $v \in \mathcal{C}^n$, $v \neq 0$, then there exists $A \in \mathcal{C}^{m \times n}$ such that $Av = u$, and $\bar{\sigma}(A) = \frac{\|u\|}{\|v\|}$.

Remark: If $Av = u$, then $\|A\| = \bar{\sigma}(A) \geq \frac{\|u\|}{\|v\|}$, so this lemma is asserting that an A of the minimal possible norm does exist which maps v onto u. This will find application in this paper in demonstrating the existence of model validating blocks Δ_i of minimal possible norm.

Proof of Lemma 3:

If $u = 0$, then take $A = 0$. Now assume $u \neq 0$. Let V be a unitary matrix whose first column is $\frac{v}{\|v\|}$. V can be constructed by starting with a matrix whose first column is $\frac{v}{\|v\|}$ and the other $n-1$ columns are $n-1$ of the standard unit vectors. By omitting the standard unit vector whose non-zero element is in the same position as a non-zero component of v, this matrix has linearly independent columns, and V may be formed by applying the Gram-Schmidt orthogonalization procedure. Let S be the n by m diagonal matrix whose first diagonal element is $\frac{\|u\|}{\|v\|}$ and the remaining diagonal elements are arbitrary real scalars which are no bigger than the first. Let U be a unitary matrix whose first column is $\frac{u}{\|u\|}$. Let $A = USV^H$. Then (U, S, V) is an SVD of A, so $\bar{\sigma}(A) = \frac{\|u\|}{\|v\|}$; and $Av = u$. □

3.1 Full Complex Blocks Only

We now work toward determining when there exist model validating $\Delta \in \mathcal{D}$. To this end, we must start with uncertainty signals ξ and η which satisfy Equation (29) and look for Δ for which Equation (27) is also satisfied. Partition ξ and η into components corresponding to the block structure of Δ so that

Equation (27) could be written as

$$\left\{ \begin{array}{c} \xi_1 \\ \xi \\ \vdots \\ \xi_\tau \end{array} \right\} = \left[\begin{array}{cccc} \Delta_1 & 0 & \cdots & 0 \\ 0 & \Delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_\tau \end{array} \right] \left\{ \begin{array}{c} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_\tau \end{array} \right\}. \quad (30)$$

Since, for each $i = 1, \ldots, \tau$, $\xi_i = \Delta_i \eta_i$, one can never have both $\xi_i \neq 0$ and $\eta_i = 0$. However, the parameterization in Equation (29) does not guarantee that these conditions never occur. To have some terminology to use to indicate that we are excluding this possibility, we make the following definition:

Definition ($\mathcal{D}$-realizable):

A signal pair (ξ, η) will be called $\mathcal{D}$-realizable if, for each $i = 1, \ldots, \tau$, either $\xi_i = 0$ or $\eta_i \neq 0$.

Satisfaction of the test in Lemma 2 allows a cancellation of the nominal output error by a combination of the fictitious signals from the uncertainty block and a norm bounded output noise while it will be shown that Lemma 3 guarantees that a structured, full complex uncertainty always exists for any $\mathcal{D}$-realizable pair (ξ, η) which satisfy Equation (29). We state an existence condition and a parameterization of all model validating uncertainty sets as follows:

Theorem 1 (structured, full complex blocks):

Suppose the conditions given in Lemma 2 are satisfied. Then, all model validating sets of uncertainty are given by

$$\mathcal{S}_{W\phi\psi} := \{\Delta \in \mathcal{D} : \Delta = W\Delta_B, \bar{\sigma}(\Delta_B) \leq 1\}, \quad (31)$$

where

$\psi \in \mathcal{C}^{n_\psi}$, $\|\phi\| \leq b_o$, $W := \mathrm{diag}(w_1 I_{n_1}, \ldots, w_\tau I_{n_\tau})$ is any diagonal complex matrix whose diagonal elements satisfy

$$|w_i| \geq \frac{\|\xi_i\|}{\|\eta_i\|}, \quad i = 1, \ldots, \tau, \quad (32)$$

and the (ξ, η) pair parameterized by ϕ and ψ as given in Equation (29) is $\mathcal{D}$-realizable.

Proof of Theorem 1:

It is first demonstrated that each $\mathcal{S}_{W\phi\psi}$ described in the statement of Theorem 1 is a model validating set. Since the conditions of Lemma 2 are satisfied, the ξ given by Equation (29) combines with a noise ν for which $\|\nu\| \leq 1$ to produce $e_y = 0$. Therefore, $\mathcal{S}_{W\phi\psi}$ will be model validating if there exists $\Delta \in \mathcal{S}_{W\phi\psi}$ for which Equation (30) is satisfied. We construct this Δ block by block. If $\xi_i = 0$, then Δ_i and Δ_{Bi} may be taken to be 0 and $\bar{\sigma}(\Delta_{Bi}) = 0$. If $\xi_i \neq 0$, then since (ξ, η) is a $\mathcal{D}$-realizable pair, $\eta_i \neq 0$ and by Lemma 3, there exists a Δ_i with $\Delta_i \eta_i = \xi_i$ and $\bar{\sigma}(\Delta_i) = \|\xi_i\|/\|\eta_i\|$. This means that if $w_i \neq 0$ and $\Delta_{Bi} := (1/w_i)\Delta_i$, then $\bar{\sigma}(\Delta_{Bi}) \leq 1$ (if $w_i = 0$, Δ_{Bi} may be chosen arbitrarily with $\bar{\sigma}(\Delta_{Bi}) \leq 1$). It follows that if

$$\Delta_B := \mathrm{diag}(\Delta_{B1}, \Delta_{B2}, \cdots, \Delta_{B\tau}),$$

then $\bar{\sigma}(\Delta_B) \leq 1$. Then $\Delta = W\Delta_B \in \mathcal{S}_{W\phi\psi}$, so $\mathcal{S}_{W\phi\psi}$ is model validating.

Now let $\mathcal{S}_W$ be an arbitrary model validating set. Let Δ be a model in $\mathcal{S}_W$ which zeros the error to within a noise level of magnitude bounded by 1. Then there must exist signals ξ and η which satisfy the P – Δ feedback conditions in Equations (27) and (28) such that ξ zeros out the error with some noise signal ν of norm no more than 1. Then by Lemma 2, there must exist ϕ satisfying condition (24) and ψ such that ξ and η are expressed in terms of ϕ and ψ by Equation (29). Since $\xi_i = \Delta_i \eta_i$ for all $i = 1, \ldots, \tau$, the pair (ξ, η) must be $\mathcal{D}$-realizable, and

$$\|\xi_i\| = \|\Delta_i \eta_i\| \leq \bar{\sigma}(\Delta_i)\|\eta_i\| \leq |w_i|\|\eta_i\|,$$

the last inequality holding because $\Delta \in \mathcal{S}_W$. This establishes that $\mathcal{S}_W = \mathcal{S}_{W\phi\psi}$. □

Note that from the remark following Lemma 3, for fixed parameters ϕ and ψ, the smallest model validating uncertainty set (as measured by the magnitude of the uncertainty weight w_i) is given by $|w_i| = \frac{\|\xi_i\|}{\|\eta_i\|}, i = 1, \ldots, \tau$. This is the minimum norm model validating result reported in [7].

Note that Lemma 2 can be viewed as a necessary and sufficient condition for the existence of a model validating set for an LFT with only structured full complex blocks. The issue that remains is how large must the uncertainty size be for model validation, which is addressed in Theorem 1.

From Theorem 1, note that any choice of uncertainty set $\{w_i, i = 1, \ldots, \tau\}$ such that

$$|w_i| < \inf_{\substack{\psi, \phi \\ \|\phi\| \leq b_o}} \frac{\|\xi_i\|}{\|\eta_i\|} \qquad \text{for some } i \quad (33)$$

will not be a model validating set. However, the converse is not necessarily true. The *infimum* might not actually be achieved by any choice of ϕ and ψ, and the values of ϕ and ψ which give a good bound on w_i for one value of i might not be the same as values which give good bounds for other values of i.

3.2 With Repeated Scalar Blocks

For a more general uncertainty structure which includes repeated scalar blocks, we assume for convenience, that all repeated scalar blocks are grouped into the first r blocks in Δ. Then, Equation (30) becomes

$$\left\{ \begin{array}{c} \xi_1 \\ \vdots \\ \xi_r \\ \xi_{r+1} \\ \vdots \\ \xi_\tau \end{array} \right\} = \Delta \left\{ \begin{array}{c} \eta_1 \\ \vdots \\ \eta_r \\ \eta_{r+1} \\ \vdots \\ \eta_\tau \end{array} \right\}, \text{ where} \tag{34}$$

$$\Delta = \mathrm{diag}(\delta_1 I_{n_1}, \ldots, \delta_r I_{n_r}, \Delta_{r+1}, \ldots, \Delta_\tau). \tag{35}$$

A further restriction is imposed that $\delta_i \in \mathcal{F}_i$, $i = 1, \ldots, r$ where $\mathcal{F}_i$ is either the field $\mathcal{R}$ of real numbers or the field $\mathcal{C}$ of complex numbers at the designer's choosing. Since Δ contains repeated scalar blocks, the condition in Lemma 2 or Theorem 1 becomes only a necessary condition for model validation. So, with repeated scalar blocks, we ask whether, among all $\mathcal{D}$-realizable pairs (ξ, η) satisfying Equation (29) with ϕ subject to the norm condition (24), a pair exists for which a Δ of the form given in Equation (35) also exists so that Equation (34) is satisfied?

First, note that given any such $\mathcal{D}$-realizable pair (ξ, η), Δ_{r+1}, ..., Δ_τ always exist by Lemma 3, so we need to consider only the existence of the first r blocks, $\delta_1 I_{n_1}$, ..., $\delta_r I_{n_r}$ where $\delta_i \in \mathcal{F}_i$. Let us denote $\bar{n} := \sum_{i=1}^r n_i = \bar{m} := \sum_{i=1}^r m_i$. A model validating uncertainty set exists for a system with repeated scalar block if and only if there exists $\delta_i \in \mathcal{F}_i$, $i = 1, \ldots, r$, ψ, ϕ, $\|\phi\| \leq b_o$ such that

$$\xi_i = \delta_i \eta_i, \quad i = 1, \ldots, r \tag{36}$$

where from Equation (29):

$$\xi_i = \xi_{o,i} + [A, \quad B]_i \left\{ \begin{array}{c} \phi \\ \psi \end{array} \right\} \tag{37}$$

$$\eta_i = \eta_{o,i} + P_{11,i}[A, \quad B] \left\{ \begin{array}{c} \phi \\ \psi \end{array} \right\} \tag{38}$$

The subscript i indicates that the correct blocks of rows have been selected for Equations (37) and (38) to make sense in the context of Equation (29) and the decompositions of ξ and η give in Equation (34).

The condition in Equation (36) can be seen as a collinearity condition in the vector space $\mathcal{C}^{n_i}$ with coefficients from the field $\mathcal{F}_i$. Consequently, a measure of distance between two subspaces can be used (see for example [13])

$$\mathrm{dist}^{(\mathcal{F}_i)}(\xi_i, \eta_i) := \|P_{\xi_i}^{(\mathcal{F}_i)} - P_{\eta_i}^{(\mathcal{F}_i)}\| \tag{39}$$

where $P_{\xi_i}^{(\mathcal{F}_i)}$ and $P_{\eta_i}^{(\mathcal{F}_i)}$ denote orthogonal projections onto the subspaces spanned over the field $\mathcal{F}_i$ by the single vectors ξ_i and η_i, respectively. We summarize our results as follows:

Theorem 2 (with repeated scalar block):

(a) Suppose the test in Lemma 2 passes. Then a model validating set exists with $\Delta_i = \delta_i I_{n_i}$, $i = 1, \ldots, r$, $\delta_i \in \mathcal{F}_i$ if and only if there exists ψ, and ϕ with $\|\phi\| \leq b_o$ such that the (ξ, η) pair parameterized by ϕ and ψ as given in Equation (29) is $\mathcal{D}$-realizable and

$$\begin{array}{ll} \text{for each} & i = 1, \ldots, r \\ \text{either} & \xi_i = 0 \\ \text{or} & \mathrm{dist}^{(\mathcal{F}_i)}(\xi_i, \eta_i) = 0 \end{array} \tag{40}$$

where ξ_i and η_i are given by Equations (37) and (38).

(b) Furthermore, if a model validating set exists, then all such sets are given by

$$\mathcal{S}_{W\phi\psi} := \{\Delta \in \mathcal{D} : \Delta = W\Delta_B, \bar{\sigma}(\Delta_B) \leq 1\}, \tag{41}$$

where

$\psi \in \mathcal{C}^{n_\psi}$, $\|\phi\| \leq b_o$, $W := \mathrm{diag}(w_1 I_{n_1}, \ldots, w_\tau I_{n_\tau})$ is any diagonal complex matrix whose diagonal elements satisfy

$$|w_i| \geq \frac{\|\xi_i\|}{\|\eta_i\|}, \quad i = 1, \ldots, \tau, \tag{42}$$

and the (ξ, η) pair parameterized by ϕ and ψ as given in Equation (29) is $\mathcal{D}$-realizable and satisfies condition (40).

In order to prove Theorem 2, we first introduce a lemma.

Lemma 4:

For each fixed i, condition (40) holds if and only if there exists a $\delta_i \in \mathcal{F}_i$ such that $\xi_i = \delta_i \eta_i$.

Proof of Lemma 4:

Fix i. First, assume that condition (40) holds for this i. If $\xi_i = 0$, then δ_i may be chosen to be 0. If $\xi_i \neq 0$, then, since $\mathrm{dist}^{(\mathcal{F}_i)}(\xi_i, \eta_i) = 0$, then the subspaces spanned over the field $\mathcal{F}_i$ by the single vectors ξ_i and η_i are the same, so there must exist $\delta_i \in \mathcal{F}_i$ for which $\xi_i = \delta_i \eta_i$.

Now assume that there exists a $\delta_i \in \mathcal{F}_i$ such that $\xi_i = \delta_i \eta_i$. If either δ_i or η_i is 0, then ξ_i is also 0 and condition (40) holds. Assuming that neither δ_i nor η_i is 0, then ξ_i is also non-zero, so ξ_i and η_i span the same subspace of the vector space $\mathcal{C}^{n_i}$ over the field $\mathcal{F}_i$. It follows that $\mathrm{dist}^{(\mathcal{F}_i)}(\xi_i, \eta_i) = 0$. $\square$

Proof of Theorem 2:

First, suppose that there exist ψ and ϕ with $\|\phi\| \leq b_o$ such that the (ξ, η) pair parameterized by

ϕ and ψ as given in Equation (29) is $\mathcal{D}$-realizable and condition (40) holds. Then by Theorem 1, any set $\mathcal{S}_{W\phi\psi}$ as given in Equation (31) with W satisfying condition (32) is model validating. Let $\Delta \in \mathcal{S}_{W\phi\psi}$ be a model validating model. The key properties of Δ are that, for each $i = 1, \ldots, \tau$, $\|\Delta_i\| \leq |w_i|$ and $\xi_i = \Delta_i \eta_i$. Because of condition (40), Lemma 4 tells us that we can replace each Δ_i for $i = 1, \ldots, r$ by a matrix of the form $\delta_i I_{n_i}$ with $\delta_i \in \mathcal{F}_i$ and still have $\xi_i = \Delta_i \eta_i$. Also, since $\xi_i = \delta_i \eta_i$ and W satisfies condition (32), we still have $\|\Delta_i\| \leq |w_i|$ with this new Δ_i. This establishes the "if" part of Theorem 2(a), and shows that every model validating set with the specified repeated scalar blocks has the form shown in Theorem 2(b).

Now suppose that a model validating set $\mathcal{S}$ exists which contains a model validating model Δ with $\Delta_i = \delta_i I_{n_i}$, $i = 1, \ldots, r$, $\delta_i \in \mathcal{F}_i$. Theorem 1 tells us that $\mathcal{S}$ must satisfy all of the conclusions of Theorem 2(b) except the condition (40). However, since for each $i = 1, \ldots, r$, $\Delta_i = \delta_i I_{n_i}$, it follows from Lemma 4 that condition (40) is satisfied. This completes the proof that every model validating set falls under the description given in Theorem 2(b), and completes the "only if" part of the proof of Theorem 2(a). $\square$

4 Uncertainty Bound Optimizations

With the parameterization of all model validating uncertainty sets given by Theorems 1 and 2, a controls engineer still faces the issue of what to do with the remaining freedom. Specifically, one may ask: can we find an "optimal" set from the given parameterization of all model validating uncertainty sets ? A more basic issue is: what is or what should an "optimal" model validating set be?

For a single uncertainty block problem, a smallest-norm model validating uncertainty appears to be a physically reasonable uncertainty bound based on Ockham's razor argument in modeling physical systems. However, for problems with a general LFT uncertainty structure, a smallest norm uncertainty bound (in a multi-objective sense) may not have any concrete physical justification. This is because for problems with multiple uncertainty blocks, their relative numerical values may not necessarily indicate their relative physical significance. For example, in robust stability [14], the determination of whether a controller guaranteeing robust stability exists or not may depend more strongly on the distribution of the uncertainty bounds over a given set of uncertainty components than on the size of largest uncertainty component.

In this section, we outline two algorithms based on constrained nonlinear optimization to determine an uncertainty model which is, in some sense, "optimal". One starting point for an optimization would be for the designer to select a nominal plant model, P, a matrix, $V = \mathrm{diag}(v_1, \ldots, v_{n_y})$, of bounds on the noise levels, and a matrix, $W = \mathrm{diag}(w_1, \ldots, w_\tau)$, of desired levels of uncertainty in the uncertainty blocks. The important feature of W is that the relative sizes of the w_i reflect the designer's desires as to the relative size of the uncertainty levels in the different blocks of a model validating $\Delta \in \mathcal{D}$.

4.1 Existence Conditions

Two conditions have been given, (17) and (21), which are necessary for the existence of model validating sets in either the case that the uncertainty structure consists of only full complex blocks (Theorem 1) or contains some repeated scalar blocks (Theorem 2). In the case of only full complex blocks, these conditions are also sufficient for existence (Theorem 1). As a prelude to trying to optimize uncertaintly levels, these necessary conditions should be checked.

As noted earlier, if all of the output channels are being modeled as having noise in them, so that the V matrix is non-singular, then the matrix M is of full rank, and condition (17) always holds. If the diagonal matrix V does have zeros on the diagonal, then a check on condition (17) can be made by first performing an SVD on M: $M = U_M \Sigma_M V_M^H$ where U_M and V_M are unitary and Σ_M is a nonnegative real diagonal matrix of the same shape as M, i.e., with more columns than rows, whose diagonal elements are in decreasing order. Condition (17) also holds if M is full rank which is equivalent to Σ_M having no zero rows. If M is rank deficient, then partition $U_M = [U_{M1}, U_{M2}]$ where the block $U_{M,1}$ corresponds to the non-zero rows of Σ_M and the block $U_{M,2}$ corresponds to the zero rows of Σ_M. Then a necessary and sufficient condition in the case of rank deficient M for condition (17) to hold is that $U_{M2} e_y^o = 0$.

To verify condition (21), the SVD of M is computed in the form:

$$M = \begin{bmatrix} U_{M1} & U_{M2} \end{bmatrix} \begin{bmatrix} \Sigma_{M1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{M1}^H \\ V_{M2}^H \end{bmatrix} \qquad (43)$$

In this decomposition, $U = [U_{M1}\ U_{M2}]$ and $V = [V_{M1}\ V_{M2}]$ are unitary, Σ_{M1} is diagonal and nonsingular, and the block matrix partitionings are conformal. Then, in Equation (18), $M^+ = V_{M1} \Sigma_{M1}^{-1} U_{M1}^H$

and N_M can be taken to be V_{M2}. Then, $(N_M)_\nu$ is decomposed as in Equation (20), and all of the components are at hand to execute the test in condition (21).

4.2 Full Complex Blocks

The idea is to normalize all uncertainties using desired levels of uncertainty in the uncertainty blocks, and then seek the smallest model validating scaled set. Specifically, we propose using nonlinear constrained optimization with ψ and ϕ as the design parameters to find a minimal positive x such that $\mathcal{S}_{(xW)}$ is a model validating set.

By Theorem 1, $\mathcal{S}_{(xW)}$ is a model validating set if there exist ψ and ϕ with $\|\phi\| \leq b_o$ such that

$$x|w_i| \geq \frac{\|\xi_i\|}{\|\eta_i\|}, \quad i = 1, \ldots, \tau \tag{44}$$

where the vectors ξ and η calculated from ψ and ϕ in Equation (29) form a $\mathcal{D}$-realizable pair. This implies that $\mathcal{S}_{(xW)} = \mathcal{S}_{(xW)\psi\phi}$. By squaring and clearing fractions, the previous inequality can be combined with the $\mathcal{D}$-realizability condition in the single inequality in Equation (46). This also has the advantage of being a polynomial in x and the components of ξ and η. The optimization problem can now be stated:

Optimal uncertainty algorithm (Full block):

$$\min_{\psi,\phi,x} x \tag{45}$$

subject to the constraints

$$\begin{aligned} \|\xi_i\|^2 - x^2|w_i|^2\|\eta_i\|^2 &\leq 0, \quad i = 1, \ldots, \tau & (46)\\ x &\geq 0 & (47)\\ \|\phi\| &\leq b_o & (48) \end{aligned}$$

where ξ_i and η_i are given by Equations (37) and (38).

For the special case where the noise is known or given, the noise can be incorporated in the e_y^o vector and the bounded parameter ϕ is not used in the parameterization of ξ and η. This is the case which is derived in the earlier minimum norm model validating solution in [7]-[9].

4.3 With Repeated Scalar Blocks

For the case with repeated scalar blocks, an optimization algorithm similar to full complex block case but with the additional collinearity condition is proposed. Similarly, by Theorem 2, $\mathcal{S}_{(xW)}$ is a model validating set if there exist ψ and ϕ with $\|\phi\| \leq b_o$ such that condition (40) is satisfied where ξ_i and η_i is a $\mathcal{D}$-realizable pair parameterized by Equations (37) and (38). Instead of using the distance condition in Equation (40) to guarantee existence, Equation (36) is used. The tradeoff is that Equation (40) leads to a quartic in the design variables while Equation (36) leads to only a quadratic at the expense of additional variables, $\delta_1, \ldots, \delta_r$. Note that the collinearity condition in Equation (36) for the set of r repeated scalar blocks and the $\mathcal{D}$-realizability condition leads to a simplification of the first r set of inequality constraints in Equation (46). The optimization problem can now be stated:

Optimal uncertainty algorithm (With Repeated Scalar block):

$$\min_{\psi,\phi,\delta_1,\ldots,\delta_r,x} x \tag{49}$$

subject to the constraints

$$\begin{aligned} \delta_i &\in \mathcal{F}_i & (50)\\ |\delta_i|^2 - x^2|w_i|^2 &\leq 0, \quad i = 1, \ldots, r & (51)\\ \|\xi_i\|^2 - x^2|w_i|^2\|\eta_i\|^2 &\leq 0, \quad i = r+1, \ldots, \tau & (52)\\ \xi_i &= \delta_i \eta_i, \quad i = 1, \ldots, r & (53)\\ x &\geq 0 & (54)\\ \|\phi\| &\leq b_o & (55) \end{aligned}$$

where ξ_i and η_i are given by Equations (37) and (38).

Remark: In Section 4.1, tests were given which were necessary, but not sufficient for the existence of a model validating set for the case that some uncertainty blocks have repeated scalar blocks. Such a sufficient condition is found by the location of a feasible point in the preceeding optimization problem. In particular, in order to have a model validating set in this case, it must be necessary to satisfy constraints (50) and (53).

To summarize, the above optimization algorithm has various physical significances. The cost in Equation (49) represents a positive scaling factor of the normalized (by user provided desirable weights) uncertainty norm bounds for each component. Inequalities (51) and (52) represents the scaled bounds on r repeated scalar uncertainties and the $\tau - r$ non-repeated full complex uncertainty bounds, respectively. A violation of these inequalities implies that xw_i is not an upper bound on the ratio of signal norms, i.e., it fails as an uncertainty bound. Clearly, inequalities (51) and (52) will more likely be satisfied with larger weights xw_i, which makes intuitive sense. The collinearity condition in Equation (53) represents the necessary structural constraints due to the repeated scalar uncertainties. Inequality (54) is the non-negative condition on the uncertainty scale factor. Note that $x \geq 1$ indicates that the current scaling makes the uncertianty bounds larger or equals to the *a priori* target while $x < 1$ indicates that there exists a smaller (for every component) model validating set than the *a priori* target. Finally, inequality (55) represents the limited freedom available as an "admissible" noise.

Consider an important special case where the repeated scalar blocks arises due to parametric uncertainties which are independent of frequencies. To reflect this condition in applications, one may wish to specify a fixed bound (i.e., a parametric error allowance) for the parametric uncertainties. This can easily be done by eliminating the scale factor x in the r inequalites in Equation (51). The optimization problem then tries to solve for the smallest non-parametric/unmodeled dynamics uncertainty subject to bounded noise and parametric uncertainties.

5 Concluding Remarks

For models of physical systems where the uncertainty is described by a linear fractional transformation with all unknown but bounded exogenous disturbance occuring at the measured outputs, feasibility conditions for model validation are derived. For the case with only structured full complex blocks, the feasiblity condition can be readily tested and involves only numerically easy constant matrix checks at all discrete frequencies. The feasibility test is actually a test on the richness of the *a priori* uncertainty structure with some help from a frequency weighted unknown but bounded measurement noise allowance. We have shown that for the case with only structured full complex blocks, this feasibility condition is also necessary and sufficient condition for the existence of a model validating uncertainty set. It is significant that this necessary and sufficient conditions hold for an arbitrary number of full complex blocks. For the more general case when repeated scalar uncertainties are also present, we have shown that an additional condition involving a collinearity test is required for model validation feasibility.

The feasibility test has a binary outcome. If it fails, it indicates either a lack of richness of the *a priori* uncertainty structure (because no bounds are assumed for the structured uncertainties) and/or that the frequency weighted 2-norm measurement noise allowance is too small. If the test passes, we have shown how all model validating uncertainty sets in $\mathcal{D}$ for the nominal plant P and the given noise level V can be parameterized. Changing any or all of $\mathcal{D}$, P, and V would produce a parameteriation of additional model validating sets of uncertainty. Obviously, model validating sets of uncertainties are in general underdetermined so that a useful general modeling/design tool should easily span the large subset of model validating uncertainties. For the above reason, we formulate two optimization algorithms based on two metrics of what is basically a multi-objective problem.

Preliminary application results are encouraging (see the illustrative example in the recent work [11]) but further theoretical refinements, algorithmic improvements, and validation through actual application on laboratory testbeds of the proposed methodology is needed.

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